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CONCERNING THE EQUATIONS OF THE ACOUSTICS OF MOVING MEDIA

A. I. Shnip

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Using the methods of rational mechanics, a new equation of the acoustics of inhomogeneous unsteadily moving media is derived. Its advantage over traditional approaches is demonstrated.

1. Equations of the acoustics of an inhomogeneous moving medium were constructed in the classical works by Blokhintsev [1]. They are obtained within the framework of the theory of perturbations by linearizing the equations of thermohydrodynamics for small perturbations of a given (unperturbed) solution of these equations that describes the unperturbed motion of the medium. The result is a system of equations for perturbations of velocity, density, and entropy. This system is complex: it contains the fields of velocity, density, and entropy of unperturbed motion as parameters and is resolved, as a rule, only for comparatively simple particular cases. At the same time it is known that linear approximations for systems of nonlinear equations may differ and correspondingly be more or less convenient and also more or less adequate for the purposes set, depending on the stage at which linearization is carries out. For example, in [2], to analyze the stability of the process of drawing fibers, the system of nonlinear equations that describe the drawing of fibers from viscous liquid was linearized relative to small perturbations not directly, as it was usually done, but after its reduction, by special transformation, to one equation. This approach made it possible to obtain a simple formulation of the problem that has a number of additional advantages.

A similar approach is used in the present work to derive equations of the acoustics of moving media. In contrast to the traditional approach, here, using the methods of rational mechanics [4], the equation of motion is linearized in the reference (Lagrangian) representation and then converted to a spatial (Eulerian) representation, retaining only terms linear in perturbations. As a result, for the case of barotropic perturbation one equation is obtained for the single quantity, viz., the field of the displacement of the particles of the medium relative to their position in unperturbed motion. In the concluding part the advantages of the equation obtained over the traditional approach are discussed. Everywhere in what follows definitions and notation of tensors and tensor operations are used, and also the apparatus of the kinematics of solid media adopted in the book by Trusdell [3]. The bold upper- and lower-case letters **X** denote points in the three-dimensional point Euclidean space and vectors in its translational space, while the bold upper-case letters denote second-rank tensors. The vector variables symbolized by Greek letters are denoted by arrows.

2. Let us consider a medium that performs motion described by the deformation function $\vec{\chi}_\kappa$ relative to the reference configuration κ ([3], II para 4):

$$\mathbf{x} = \vec{\chi}_\kappa (\mathbf{X}, t) , \tag{1}$$

where **x** is the position (the point of the point Euclidean space) at the time instant t of that material particle which in the reference configuration κ was at the **X** point.

The motion of the medium (in the absence of mass forces) obeys the first Cauchy law ([3], III para 5), i.e., the momentum conservation law:

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T} , \tag{2}$$

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where ρ is the density; T is the Cauchy stress tensor; v is the velocity of the medium defined as

$$\mathrm{v}(\mathbf{X}, t) = \frac{\partial}{\partial t}\vec{\chi}_\kappa(\mathbf{X}, t) = \dot{\vec{\chi}}_\kappa(\mathbf{X}, t). \tag{3}$$

The dot over the symbol denotes the time derivative in the reference representation, i.e.,

$$\dot{\mathrm{v}} = \frac{\partial}{\partial t}\mathrm{v}(\mathbf{X}, t) = \frac{\partial^2}{\partial t^2}\vec{\chi}_\kappa(\mathbf{X}, t) = \ddot{\vec{\chi}}_\kappa. \tag{4}$$

We assume that the motion (deformation) is represented as a superposition of two components: background, i.e., unperturbed motion, and acoustic perturbation. Thus:

$$\vec{\chi}_\kappa(\mathbf{X}, t) = \vec{\chi}^0(\mathbf{X}, t) + \vec{\xi}(\mathbf{X}, t), \tag{5}$$

where $\vec{\chi}^0$ is the background motion; $\vec{\xi}$ is the perturbation, which is assumed to be small. In physical meaning the vector $\vec{\xi}$ is the displacement of the material particles of the medium relative to their position in unperturbed motion. According to what has been said above, a similar division can also be made for the density and the stress tensor, and we will assume that perturbations of the stress tensor have the form of isotropic pressure (this corresponds to the standard assumption that the shear viscosity is negligible in acoustic motion). Consequently, we have

$$\mathbf{T} = \mathbf{T}^0 - p'\mathbf{I} = \mathbf{T}_0 - p_0\mathbf{I} - p'\mathbf{I}, \tag{6}$$

$$\rho = \rho_0 + \rho', \tag{7}$$

where $\mathbf{T}^0$ is the stress tensor in background motion; p' is the acoustic pressure ($p'\mathbf{I}$ is the perturbation of the stress tensor; $\mathbf{I}$ is a unit tensor; p_0 is the pressure in background motion; $p_0\mathbf{I}$ is the equilibrium portion of the stress tensor in background motion; $\mathbf{T}_0 = \mathbf{T}^0 + p_0\mathbf{I}$ is the nonequilibrium portion of the stress tensor in background motion; ρ_0 is the density in background motion; ρ' is the perturbation of density.

We will emphasize that here ρ_0 is a function of $\mathbf{X}$ and t just as are all the remaining quantities, i.e., the background motion, generally speaking, is inhomogeneous, nonstationary, and compressible.

As usual, we assume the existence of an equation of state that prescribes the dependence of pressure on the temperature ϑ and density ρ:

$$p = \hat{p}(\vartheta, \rho), \tag{8}$$

so that

$$p_0 = \hat{p}(\vartheta_0, \rho_0), \tag{9}$$

$$p' = \hat{p}(\vartheta, \rho) - \hat{p}(\vartheta_0, \rho_0) = \hat{p}(\vartheta_0 + \vartheta', \rho_0 + \rho') - \hat{p}(\vartheta_0, \rho_0), \tag{10}$$

where, by analogy with (6) and (7), for the temperature ϑ the division of $\vartheta = \vartheta_0 + \vartheta'$ into the background temperature ϑ_0 and perturbation of the temperature ϑ' is adopted. We will consider the acoustic process to be barotropic, i.e., such that there exists a fractional dependence between ϑ' and ρ' prescribed, say, by the function $\tilde{\vartheta}(\rho)$ so that $\vartheta' = \tilde{\vartheta}(\rho')$ ($\tilde{\vartheta}(0) = 0$). Then we may write (here and hereafter the symbol: = means "equality by definition")

$$p' = \tilde{p}(\vartheta_0, \rho_0, \rho') := \hat{p}(\vartheta_0 + \tilde{\vartheta}(\rho'), \rho_0 + \rho') - \hat{p}(\vartheta_0, \rho_0). \tag{11}$$

Isothermal or adiabatic processes can serve as examples of barotropic processes. Linearizing in (11) the function $\tilde{p}$ relative to the small value ρ', we obtain

$$p' = \frac{\partial \tilde{p}(\vartheta_0, \rho_0, 0)}{\partial \rho'} \rho' : = a^2 \rho' , \tag{12}$$

where a (the speed of sound) is defined as

$$a(\vartheta_0, \rho_0) : = \sqrt{\left(\frac{\partial \tilde{p}(\vartheta_0, \rho_0, 0)}{\partial \rho'}\right)} = \sqrt{\left(\frac{\partial \tilde{p}(\vartheta_0, \rho_0, 0)}{\partial \vartheta_0} \frac{\partial \tilde{\vartheta}(0)}{\partial \rho'} + \frac{\partial \hat{p}(\vartheta_0, \rho_0)}{\partial \rho_0}\right)} . \tag{13}$$

Since in the absence of perturbations the small additions $\vec{\xi}$, p', ρ' disappear, and the momentum balance must be satisfied, then

$$\rho_0 \ddot{\vec{\chi}}^0 = \operatorname{div} \mathbf{T}^0 . \tag{14}$$

Let us turn to the continuity equation presented in the reference description ([3], II para 5):

$$\rho_r = \rho \det \mathbf{F} , \tag{15}$$

where ρ_r is the density in the reference configuration (it is assumed that the reference configuration is such that ρ_r = const; F is the tensor of the deformation gradient defined as

$$\mathbf{F} = \nabla_{\mathbf{x}} \vec{\chi}_\kappa . \tag{16}$$

With account for (5), expression (16) can be represented as

$$\mathbf{F} = \mathbf{F}^0 + \mathbf{F}' , \tag{17}$$

where $\mathbf{F}^0 = \nabla_{\mathbf{x}} \vec{\chi}^0$; $\mathbf{F}^0 = \nabla_{\mathbf{x}} \vec{\xi}$.

Having written the continuity equation (15) for background motion, from it we find

$$\det \mathbf{F}^0 = \frac{\rho_r}{\rho_0} . \tag{18}$$

We linearize (15) with account for (5) relative to the perturbation $\vec{\xi}$ (see [3], the solution of exercise II 5.1)

$$\rho = \frac{\rho_r}{\det(\mathbf{F}_0 + \mathbf{F}^\xi)} = \frac{\rho_r}{\det \mathbf{F}_0} - \frac{\rho_r}{(\det \mathbf{F}_0)^2} \operatorname{tr}(\mathbf{F}^\xi \operatorname{adj} \mathbf{F}^0) = \frac{\rho_r}{\det \mathbf{F}^0} (1 - \operatorname{tr}(\mathbf{F}^\xi (\mathbf{F}^0)^{-1})) = \rho_0 + \rho' . \tag{19}$$

From this, with account for (18)

$$\rho' = -\rho_0 \operatorname{tr}(\mathbf{F}^\xi (\mathbf{F}^0)^{-1}) . \tag{20}$$

Let us go over in (20) from the reference to the spatial description. The functions represented in the spatial description will be denoted by a tilde. For the transition indicated we introduce the function $\tilde{\mathbf{X}}$, the reciprocal of $\vec{\chi}_\kappa$, i.e.,

$$\tilde{\mathbf{X}}(\mathbf{x}, t) : = \vec{\chi}_\kappa^{-1}(\mathbf{x}, t) . \tag{21}$$

The function $\tilde{\mathbf{X}}$ gives the position of $\mathbf{X}$ in the reference configuration of that material particle which at the time t was at the spatial point $\mathbf{x}$:

$$\mathbf{X} = \tilde{\mathbf{X}}(\mathbf{x}, t) . \tag{22}$$

We will define the function $\tilde{\vec{\xi}}$ as

$$\tilde{\vec{\xi}}(\mathbf{x}, t) := \vec{\xi}(\tilde{\mathbf{X}}(\mathbf{x}, t), t). \tag{23}$$

Since $\tilde{\mathbf{X}}$ is the reciprocal function of $\vec{\chi}_\kappa$, then

$$\vec{\xi}(\mathbf{X}, t) = \tilde{\vec{\xi}}(\chi_\kappa(\mathbf{X}, t), t). \tag{24}$$

We will calculate $\mathbf{F}^\xi$ using $\tilde{\vec{\xi}}$. Here, we take into account representation (5) and discard terms of the second order in $\vec{\xi}$:

$$\mathbf{F}^\xi(\mathbf{X}, t) = \nabla_\mathbf{X} \vec{\xi}(\mathbf{X}, t) = \nabla_\mathbf{X} \tilde{\vec{\xi}}(\vec{\chi}_\kappa(\mathbf{X}, t), t) = \nabla_\mathbf{x} \tilde{\vec{\xi}}(\mathbf{x}, t) \nabla_\mathbf{X} (\vec{\chi}^0(\mathbf{X}, t)) = (\mathrm{grad}\, \vec{\xi})\, \mathbf{F}^0. \tag{25}$$

In the last term the tilde is omitted, since grad means the gradient in the spatial representation ([3], II para 6). Having used (25) in (20), we obtain

$$\rho' = -\rho_0 \,\mathrm{tr}\,(\mathrm{grad}\, \vec{\xi}) = -\rho_0 \,\mathrm{div}\, \vec{\xi}. \tag{26}$$

Substituting representations (5)-(7) with account for (26), (12) and (14) into (2) and restricting ourselves to terms linear in perturbations, we obtain

$$\ddot{\vec{\xi}} - \ddot{\vec{\chi}}^0 \,\mathrm{div}\, \vec{\xi} = \frac{1}{\rho_0} \mathrm{grad}\, (\rho_0 a^2 \,\mathrm{div}\, \vec{\xi}). \tag{27}$$

By definition, the velocity in background motion ([3], II para 1) is

$$\mathbf{v}_0(\mathbf{X}, t) := \dot{\vec{\chi}}^0(\mathbf{X}, t). \tag{28}$$

Let us express this function in a spatial representation, i.e., construct the function

$$\tilde{\mathbf{v}}_0(\mathbf{x}, t) = \mathbf{v}_0(\tilde{\mathbf{X}}(\mathbf{x}, t), t) = \dot{\vec{\chi}}^0(\tilde{\mathbf{X}}(\mathbf{x}, t), t). \tag{29}$$

Since

$$\mathbf{v}_0(\mathbf{X}, t) = \tilde{\mathbf{v}}_0(\vec{\chi}_\kappa(\mathbf{X}, t), t), \tag{30}$$

then

$$\dot{\mathbf{v}}_0 \equiv \ddot{\vec{\chi}}^0 = \frac{\partial \tilde{\mathbf{v}}_0}{\partial t} + (\mathrm{grad}\, \dot{\mathbf{v}}_0)(\tilde{\mathbf{v}}_0 + \dot{\vec{\xi}}). \tag{31}$$

Now, we express $\ddot{\vec{\xi}}$ in (27) in terms of the function (29) $\tilde{\vec{\xi}}$. For this, first we calculate $\dot{\vec{\xi}}$, taking account of (5) and retaining only terms linear in $\vec{\xi}$:

$$\dot{\vec{\xi}}(\mathbf{X}, t) = \frac{d}{dt}(\tilde{\vec{\xi}}(\vec{\chi}_\kappa(\mathbf{X}, t), t)) = \nabla_\mathbf{x} \tilde{\vec{\xi}}(\vec{\chi}_\kappa(\mathbf{X}, t), t)\, \mathbf{v}_0 + \partial_t \tilde{\vec{\xi}}(\vec{\chi}_\kappa(\mathbf{X}, t), t). \tag{32}$$

Here and hereafter ∂_t means a partial derivative of the function $\tilde{\vec{\xi}}(\mathbf{x}, t)$ with respect to time (the second argument). Then for $\ddot{\vec{\xi}}$ with account for (30) and (31), retaining terms linear in $\vec{\xi}$, we have

$$\ddot{\vec{\xi}}(\mathbf{X}, t) = \frac{d}{dt} [\nabla_\mathbf{x} \tilde{\vec{\xi}}(\vec{\chi}_\kappa(\mathbf{X}, t), t)\, \mathbf{v}_0(\mathbf{X}, t) + \partial_t \tilde{\vec{\xi}}(\vec{\chi}_\kappa(\mathbf{X}, t), t)] =$$

$$= [\nabla_\mathbf{x} \nabla_\mathbf{x} \tilde{\vec{\xi}}(\vec{\chi}_\kappa(\mathbf{X}, t), t)\, \mathbf{v}_0(\mathbf{X}, t) + \partial_t \nabla_\mathbf{x} \tilde{\vec{\xi}}(\vec{\chi}_\kappa(\mathbf{X}, t), t)]\, \mathbf{v}_0(\mathbf{X}, t) +$$

$$+ \nabla_\mathbf{x} \tilde{\vec{\xi}}(\vec{\chi}_\kappa(\mathbf{X}, t), t)\, \partial_t \mathbf{v}_0(\mathbf{X}, t) + \partial_t \nabla_\mathbf{x} \tilde{\vec{\xi}}(\vec{\chi}_\kappa(\mathbf{X}, t), t)\, \mathbf{v}_0(\mathbf{X}, t). \tag{33}$$

Moving to a spatial description by means of (22), substituting the resulting expression together with (31) and (29) into (27), discarding terms quadratic in $\vec{\xi}$, and omitting the tilde, we obtain

$$\frac{\partial^2\vec{\xi}}{\partial t^2} + 2\left(\frac{\partial}{\partial t}\,\mathrm{grad}\,\vec{\xi}\right)\mathbf{v}_0 + (\mathrm{grad}\,(\mathrm{grad}\,\vec{\xi}))\,(\mathbf{v}_0 \otimes \mathbf{v}_0)\,(\mathrm{grad}\,\vec{\xi} - \mathbf{I}\,\mathrm{div}\,\vec{\xi})\left(\frac{\partial^2\mathbf{v}_0}{\partial t} + (\mathrm{grad}\,\mathbf{v}_0)\,\mathbf{v}_0\right) =$$

$$= \frac{1}{\rho_0}\,\mathrm{grad}\,(\rho_0 a^2\,\mathrm{div}\,\vec{\xi})\,, \tag{34}$$

where the symbol $\otimes$ indicates the tensor product of vectors (see [3], p. 504), and all the variables are functions of $\mathbf{x}$ and t. This is the desired equation of the acoustics of moving media.

In Cartesian coordinates Eq. (34) looks like

$$\frac{\partial^2\xi_i}{\partial t^2} + 2\,\frac{\partial^2\xi_i}{\partial t\partial x_j}\,v_j^0 + \frac{\partial^2\xi_i}{\partial x_j\partial x_k}\,v_j^0 v_k^0 + \left(\frac{\partial\xi_i}{\partial x_j} - \frac{\partial\xi_i}{\partial x_i}\,\delta_{ij}\right)\left(\frac{\partial v_j^0}{\partial t} + \frac{\partial v_j^0}{\partial x_l}\,v_l^0\right) = \frac{1}{\rho_0}\,\frac{\partial}{\partial x_i}\left(\rho_0 a^2\,\frac{\partial\xi_j}{\partial x_j}\right). \tag{35}$$

3. Equation (34) has a number of advantages over the traditional formulation of the problem considered. So, preserving generality of statement (i.e., the case of an inhomogeneous nonuniformly and unsteadily moving medium) the problem in the approximation of a barotropic process was reduced to one equation for one variable, viz., the displacement $\vec{\xi}$ of the particles of the medium relative to their position in background motion. This equation contains only the fields of velocity and density of background motion. Even in the barotropic approximation the traditional formulation is reduced to a system of two equations for the perturbations of velocity and density, which can be reduced to one equation only for vortex-free fields of background velocity and sound velocity. Moreover, in addition to velocity and density fields this system contains a pressure field. These special features of the system complicate the statement and solution of specific problems and makes the formulation of boundary conditions more difficult. The equation obtained above is free of those drawbacks. Another positive feature of the formulation suggested is the fact that the problem formulated turned out to be insensitive to the form of the rheological equation for background motion: the rheological equation can in principle correspond to a nonlinear viscous viscoelastic or even elastic medium; the sole requirement is that the approximation adopted in the derivation for perturbation of the stress tensor be a good approximation for this equation.

Still another argument in favor of the approach suggested is the fact that potentially the region of the applicability of the equation obtained is wider. In fact, the rate of acoustic vibrations in intense sound waves amounts to a value of the order of meters per second, which often can be comparable with the velocity of background motion. At the same time the displacement of the particles of the medium in a sound wave is always much smaller than the scales of the problem, in particular, of the sound wavelength. Consequently, the condition of applicability of the theory of perturbations, namely, the smallness of the perturbing parameters as compared with the background ones, is less rigid in our approach.

We can illustrate the above by an example showing the simplicity of obtaining particular cases from the general equation. For $\rho_0 a^2$ = const and the spatially uniform background velocity $\mathbf{v}_0$, by applying the divergence operation to Eq. (34) (or (35), which is more illustrative) and taking account of (26) we obtain for the scalar variable

$$\Psi = -\,\mathrm{div}\,\vec{\xi} = \frac{\rho'}{\rho_0} = \frac{1}{\rho_0 a^2}\,p'$$

the following equation

$$\frac{\partial^2\Psi}{\partial t^2} + 2\,\frac{\partial}{\partial t}\,(\mathrm{grad}\,\Psi)\cdot\mathbf{v}_0 + \mathrm{tr}\,((\mathrm{grad}\,(\mathrm{grad}\,\Psi))\,(\mathbf{v}_0 \otimes \mathbf{v}_0)) = a^2\Delta\Psi\,. \tag{36}$$

We emphasize that the terms with the factor $\partial v_0/\partial t$ drop out of the equation, since the two terms that contain this factor (the last term on the right-hand side of Eq. (35)) are mutually balanced out in this case. Equation (36) coincides completely in form with Eq. (1.85) in [1] (written, of course, for another scalar variable). The derivation of Eq. (1.85) in [1] is far from being as simple and obvious as was the case with Eq. (36). Moreover, the first was obtained only for the case of constant velocity v_0, while the other is valid also for a uniform but time-variable velocity of background motion. In [1] an attempt was made to obtain the equation of acoustics also for the latter case. However, it is derived not from the general system, but by going over, for an ordinary wave equation in a quiescent medium, to a coordinate system moving with a nonuniform velocity. As a result, we obtain an equation (see [1] (3.7)) that differs from Eq. (36) by the presence of a term of the form

$$(\text{grad}\ \Psi) \cdot \frac{\partial \mathbf{v}_0}{\partial t}. \tag{37}$$

But this derivation does not take account of the presence of inertial forces that are bound to appear in a nonuniformly moving reference system and that influence the perturbations of density; this must lead to the presence of additional terms. Having analyzed the derivation of Eq. (34), we can easily see that the term of the form $(\text{grad}\ \vec{\xi})\ \frac{d\mathbf{v}_0}{dt}$ corresponds to the term (37) in Eq. (34), whereas the term $-\text{div}\ \vec{\xi}\frac{d\mathbf{v}_0}{dt}$ just corresponds to the inertial forces associated with the perturbations of density in the reference system cocurrent with background motion. As was already emphasized above, after taking the divergence in the case of uniform velocity of background motion these two terms are mutually cancelled. Thus, relation (36) (in contrast to (3.7') in [1]) represents a correct form of the equation for the acoustics of media that move homogeneously but nonuniformly.

In conclusion we note that the above derivation of the equations of acoustics was limited, only for reasons of simplicity, to the case of barotropic approximation and can be extended in a rather routine manner to the general case by using, instead of Eq. (11), the energy equation linearized by the scheme given above. In this case the problem is reduced to a system of two differential equations for $\vec{\xi}$ and temperature perturbations.

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NOTATION

κ, reference configuration; $\vec{\chi}_\kappa$, deformation; x, point in three-dimensional Euclidean space; $\mathbf{X}$, point in space occupied by a material particle in reference configuration; ρ, density of medium; T, Cauchy stress tensor; v, velocity of medium; t, time; ξ, perturbation of deformation; p, pressure; $\mathbf{I}$, unit tensor; ϑ, temperature; $\tilde{\vartheta}(\rho)$, $\tilde{p}(\vartheta_0, \rho_0, \rho')$, functions determining the barotropic process; a, speed of sound; F, deformation gradient tensor; $\tilde{\mathbf{X}}(\mathbf{x}, t)$, reciprocal function of χ_κ; $\vec{\xi}_i$, x_j, v_j^0, Cartesian components for $\vec{\xi}$, x, v_0, respectively; Ψ, scalar acoustic function. Superscripts: 0 denotes thermomechanical quantities in background motion; ′ (prime) denotes perturbations of thermomechanical quantities; ξ denotes the portion of the deformation tensor corresponding to perturbation. Subscripts: κ identifies reference configuration for the deformation fucntion; 0 denotes pressure, density, and temperature of the background motion and also the nonequilibrium part of the stress tensor in background motion; x, X identify variables over which the gradient is taken; r, denotes density in the reference configuration; t identifies the variable (time) with respect to which the partial derivative is taken; i, j, l, k are the indices of the Cartesian components of the vectors.

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